Weakly Quasi-prime Modules and Coprime Modules
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Abstract:
Let R be a commutative ring with unity and let M be a unitary R-module. The goal of this work is to investigate the relationships between weakly Quasi-prime modules and coprime modules (dual notion of prime modules). Also we give some basic properties of the two concepts.

Keywords: prime module, coprime module, weakly Quasi-prime module.

1. Introduction
Throughout this paper all rings are commutative with identity and all modules are unital. A module M is said to be prime if for each submodule W of M, \( \text{ann}_RM = \text{ann}_RW \), where \( \text{ann}_RM = \{ r \in R : rx = 0 \text{ for each } x \in M \} \). W is said to be a weakly Quasi-prime module (briefly w.q.p) module if \( \text{ann}_RW = \text{ann}_RrW \) for every \( r \notin \text{ann}_RW \). M is a coprime R-module (dual notion of prime R-module) if and only if \( \text{ann}_RM = \text{ann}_RM \setminus N \) for each submodule \( N \neq 0 \) of M [10].

Note that \( \text{ann}_RM \setminus W = [W:M] \), equivalently M is coprime R-module if and only if M is a second module [10], where M is a second module if for every \( r \in R \), the homothety \( r^* \) on M is either zero or surjective where a homothety \( r \) on M means \( r^* \in \text{End} (M) \) and \( r^* (x) = rx \) for each x in M [10]. Ali, I.M. in [1, corollary 9] prove that M is a coprime R-module if and only if for every \( r \in R, r \neq 0 \), either \( rM = 0 \) or \( rM = M \) (that is, M is a second module).

In this work we study the relations between weakly Quasi Prime and coprime modules and give the necessary and (or) sufficient condition under which the two concepts are equivalent.

Remarks and Examples (2-1)
1-Z as Z-module is not coprime, see [1], but it is a W.q.p module, [6].
2-For all \( n, m \in Z : n \neq m \), the Z-module \( M = Z_n \bigoplus Z_m \) is not coprime, [1].
3-The Z-module \( M = Z \bigoplus Z_p \); p is prime number, is a W.q.p module, [6].
4-Every simple R-module is a W.q.p module and a coprime R-module but the convers is not true for example: Q as a Z-module is coprime [1] and w.q.p, but not simple.
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5-M as R-module is coprime if and only if \( \text{ann}_R M \) is a prime ideal,[9].

Anderson in [3], defined a divisible R-module M as : M is called divisible if for all \( r \neq 0 \) in R \( :rM=M, [3] \).
A module M is called multiplication if and only if there exist an ideal B of R for any submodule W of M, such that BM=W[8].
6-If M is a multiplication coprime, then M is a w.q.p module.
proof: since M is coprime multiplication module, then M is simple and hence M is prime[10], this implies M is quasi-prime[5], so M is a w.q.p module[6].

3.The Main Results
Theorem(3-1)
Every divisible coprime module is a w.q.p module.
Proof: let W be a divisible coprime module, then \( rW=W, so \text{ann}_R rW=\text{ann}_R W \). Suppose \( a \in \text{ann}_R W \) so \( aW=0, but \ aW=W \) implies \( W=0 \) which is a contradiction, so \( a \notin \text{ann}_R W \) which means W is a w.q.p module.

Theorem (3-2)
If W is a divisible R-module, then the following statements are equivalent.
1-W is coprime.
2-W is a w.q.p module.
Proof: 1\( \rightarrow \) 2 the proof follows by theorem (3-1). 2 \( \rightarrow \) 1 by [1] we must prove \( rW=W \) or \( rW=0 \). For each \( r \in R \), suppose that \( rW \neq W \) and \( rW \neq 0 \), so let \( rW = B \), for a submodule B of W, then \( r \in [B:W], which mean rW \subseteq B, so \text{ann}_R B \subseteq \text{ann}_R W, \) but W is a w.q.p module, so \( \text{ann}_R rW=\text{ann}_R W \) for each \( r \notin \text{ann}_R W \), implies \( \text{ann}_R B \subseteq \text{ann}_R W \) which is a contradiction.

The divisibility condition is necessary in theorem(3-2) for example:
Z as a Z-module is a (w.q.p) module,[6], but not coprime, see[10] where Z as a Z-module is not divisible.
In [8], an R-module M is called finitely generated (briefly f.g)
If there exist \( r_1 \) in R such that \( (1-r_1)M=0 \).

Theorem (3-3)
If K is a f.g. w.q.p R-module, then K is a coprime module.
Proof
Since K is a w.q.p R-module so \( \text{ann}_R K=\text{ann}_R rK \), \( \forall r \notin \text{ann}_R K, so rK \neq 0 \) But K is f.g., so there exist \( r_1 \in R \) such that \( (1-r_1)K=0 \), this implies \( K=rK \) which means K is coprime.

Theorem (3-4)
For a finitely generated R-module W, the following statement are equivalent:
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1-W is a w.q.p R-module.
2-W is coprime.
3-W is a prime module.

**Proof:**
1→ 2 by theorem(3-3) 2→ 1, since W is f.g so there exist r₁ ∈ R such that (1-r₁)r₁W=0, so r₁W=W, since W is coprime f.g so W is prime module. By [2], which means annₐW=annₐN for each submodule N of W.
To prove annₐW=annₐr₁W for every r₁∉ annₐW, since r₁W=W, so annₐW=annₐr₁W. Let x ∈ annₐr₁W so x₁W=0, x₁ ∈ annₐM, since W is prime module so by[5], annₐW is a prime ideal so either x₁∈ annₐW or r₁∈ annₐW. If r₁∈ annₐW implies r₁W=0, but r₁W=W so W=0, which is contradiction.

Therefore x₁∈ annₐW, so annₐr₁W⊆ annₐW which mean annₐW=annₐrw, For each r₁∉ annₐW.

2→ 3, the proof follows by [2].

3→ 1 , since W is f.g so there exist r₁ ∈ R ; (1-r₁)r₁W=0, [7].
So r₁W=W, we must show that annₐW=annₐr₁W for each r₁∉ annₐW
Since r₁W⊆ W so annₐW⊆ annₐr₁W, to prove annₐr₁W⊆ annₐW.
Let x ∈ annₐr₁W so x₁W=0, which mean x₁∈ annₐW, by [5] implies annₐW is a prime ideal so either x₁∈ annₐW or r₁∈ annₐW. If r₁∈ annₐW, so r₁W=0 which is a contradiction, since W is f.g, so x₁∈ annₐW, therefore annₐr₁W⊆ annₐW, so annₐr₁W=annₐW for each r₁∉ annₐW, so W is a w.q.p module.

**Theorem (3-4)**
If K is a cyclic R-module, then the following statements are equivalent:
1-K is a coprime R-module.
2-K is a prime R-module.
3-K is a w.q.p R-module.

**Proof:**
1→ 2, the proof follows by [1].
2→ 3, the proof follows by [6].

3→ 1, since K is a w.q.p R-module and cyclic, then K is prime by [6]. We must prove annₐN⊆ annₐK/N for each submodule N of K.
Let x∈ annₐN=annₐK, but K is a w.q.p R-module, so annₐr₁K=annₐK for each r₁∈ annₐK, so x∈ annₐK implies x₁K=0, so that x₁∈ annₐK but annₐK is prime ideal (since K is prime module by [6]), so either x∈ annₐK or r∈ annₐK, but r∉ annₐK, implies x∈ annₐK/N (since K/N ≤ M so annₐK⊆ annₐK/N).
In [2] A.M. Inaam submitted a non-torsion module, where an $R$-module $W$ over an integral domain $B$ is a non-torsion if there exist $w \in W$ such that $\text{ann}_B(w) = 0$.

**Proposition (3-5)**

Every non-torsion coprime $R$-module over an integral domain $R$ is a w.q.p module.

**Proof**

Let $M$ be a non-torsion coprime $R$-module. Let $x \in \text{ann}_R M$, so $x \in M$ implies $x = 0$. If $rM = 0$ implies $x = 0$, so $rM = M$ which mean $xM = 0$ so $x \in \text{ann}_R M$ implies $\text{ann}_R M = \text{ann}_R rM$. Suppose $r \in \text{ann}_R M$, but $M$ is non-torsion so $r = 0$ which is contradiction. So $r \notin \text{ann}_R M$.

Recall that a submodule $B$ of an $R$-module $M$ is called a direct summand of $M$ if and only if there exists a submodule $C$ of $M$ such that $M = B \oplus C$, [4].

Anderson submitted in [3] that $\text{ann}_R (U \oplus V) = \text{ann}_R U \cap \text{ann}_R V$, where $U, V$ are $R$-modules.

**Theorem (3-6)**

Let $M_1$ and $M_2$ be two coprime $R$-modules. Then $M = M_1 \oplus M_2$ is a w.q.p module.

**Proof:**

To prove $\text{ann}_R rM \subseteq \text{ann}_R M$. Let $x \in \text{ann}_R rM$ so $x \in M$ implies $x \in M_1 \oplus M_2 = (0,0)$. So $x \in M_1 = 0$ and $x \in M_2 = 0$, so $x \in \text{ann}_R M_1$ and $x \in \text{ann}_R M_2$.

But $M_1$ and $M_2$ are coprime so by [9] there exists a proper submodule $N_1$ of $M_1$ and a proper submodule $N_2$ of $M_2$ such that $x \in \text{ann}_R M_1$ and $x \in \text{ann}_R M_2$ implies $x \in N_1$ and $x \in N_2$, so $x \in [N_1 : M_1]$ and $x \in [N_2 : M_2]$ which mean $x \in \text{ann}_R M_1$ and $x \in \text{ann}_R M_2$ so $x \in \text{ann}_R M = \text{ann}_R (M_1 \oplus M_2)$, which mean $x \in \text{ann}_R M$, so $\text{ann}_R rM \subseteq \text{ann}_R M$, which implies $\text{ann}_R rM = \text{ann}_R M$. We must prove $r \notin \text{ann}_R M$. Suppose $r \in \text{ann}_R M = \text{ann}(M_1 \oplus M_2)$, so $r \in \text{ann}_R M \cap \text{ann}_R M$. But $M_1$ and $M_2$ are coprime so by [9], $r \in [N_1 : M_1]$ and $r \in [N_2 : M_2]$ implies $r \in N_1$ and $r \in N_2$, but $M_1$ and $M_2$ re coprime so $r \in \text{ann}_R M$, which mean $r \notin \text{ann}_R M$, which mean $M$ is a w.q.p module.

Next, we describe the relation between weakly quasi prime module and coprime module of fractional $R_S$-module $M_S$, where a subset $S$ of a ring $R$ is called multiplicatively closed if $1 \in S, 0 \notin S$ and $ab \in S$ for each $a, b \in S$. Let $M$ be an $R$-module and $S$ be a multiplicatively closed on $R$; $S \neq \emptyset$, $0 \notin S$. Let $R_S$ be the set of all fractionals $r/s$ where $r \in R, s \in S, M_S$ be the set of all fractional $x/s$ where $x, s \in S; x/s_1 = x/s_2$ if and only if there exist $t \in S$ such that $t(s_1 x_2 - s_2 x_1) = 0$. So we can make $M_S$ into $R_S$-module by setting
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\[ \frac{x+y}{t} = \frac{(tx+sy)}{st}, \quad \frac{r}{t} \cdot \frac{x}{s} = \frac{rx}{ts} \text{ for every } x,y \in M \text{ and } r \in R, s,t \in S. \]
So we can define the two maps \( \psi : R \to R_S; \psi(r) = r/1 \)
\[ : M \to M; \varphi(m) = \frac{m}{1} \text{ for each } m \in M. \]
\[ \varphi \text{ For each } r \in R, \]
If \( N \) is a submodule of an \( R \)-module \( M \) and \( S \) is a multiplicatively closed in \( R \), then \( N_S = \{n/s; n \in N, s \in S\} \) is a submodule of the \( R_S \)-module \( M_S \).[8]

**Theorem (3-7)**
Let \( M_S \) be a coprime \( R_S \)-module, then \( M_S \) is a w.q.p \( R_S \)-module for every multiplicatively closed set \( S \) of \( R \).

**Proof :**
Since \( \text{ann}_{R_S} M_S \subseteq \text{ann}_{R_S t} s M_S \). Let \( x/t \in \text{ann}_{R_S t} s M_S \), then \( x/t \cdot r/t_1 M_S = 0 \) implies \( xr/t_1 M_S = 0 \), but \( M_S \) is a coprime \( R_S \)-module, which means \( r/t_1 M_S = M_S \), i.e. \( r/t_1 M_S \neq 0 \), so \( r/t_1 \notin \text{ann} M_S \), \( x/t \notin \text{ann}_{R_S} M_S \). So \( M_S \) is w.q.p \( R_S \)-module.

**Conclusion:**
From this research we conclude that the condition make two module coprime and w.q.p are equivalent, which is divisible, finitely generated, cyclic, if \( M \) a non-torsion coprime \( R \)-module over integral domain \( R \), then \( M \) is w.q.p module. we conclude that if \( M_1 \), \( M_2 \) are two coprime, then \( M = M_1 \oplus M_2 \) is w.q.p module.

**Acknowledgement**
The researchers would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq) Baghdad, Iraq for its support in the present Research.

**References**
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الخلاصة:
حلقة A بديلية تحوي المحايد W وليكن M موديول حيادي، الفهد من هذا العمل هو ايجاد بعض العلاقات بين الموديولات الشبه اوليه الضعيفة والموديولات العكس اولية واعطينا بعض الخواص المهمة للمفهومين.
كلمات مفتاحية:
الموديولات الأولية،الموديولات العكس اولية،الموديولات الشبه اولية الضعيفة.